

# COUNTING ODD CYCLES IN LOCALLY DENSE GRAPHS

CHRISTIAN REIHER

**ABSTRACT.** We prove that for any given  $\varepsilon > 0$  and  $d \in [0, 1]$ , every sufficiently large  $(\varepsilon, d)$ -dense graph  $G$  contains for each odd integer  $r$  at least  $(d^r - \varepsilon)|V(G)|^r$  cycles of length  $r$ . Here,  $G$  being  $(\varepsilon, d)$ -dense means that every set  $X$  containing at least  $\varepsilon|V(G)|$  vertices spans at least  $\frac{d}{2}|X|^2$  edges, and what we really count is the number of homomorphisms from an  $r$ -cycle into  $G$ .

The result addresses a question of Y. KOHAYAKAWA, B. NAGLE, V. RÖDL, and M. SCHACHT.

## §1. INTRODUCTION

A graph  $G$  is said to be  $d$ -quasirandom for some real number  $d \in [0, 1]$  if each subset  $X$  of the set  $V$  of its vertices spans  $\frac{d}{2}|X|^2 + o(|V|^2)$  edges. A part of the reason as to why this concept is a useful one is that it is known that into any such graph there are  $(d^{|E(H)|} + o_H(1))|V(G)|^{|V(H)|}$  homomorphisms from any fixed graph  $H$ .

Recently Y. KOHAYAKAWA, B. NAGLE, V. RÖDL, and M. SCHACHT [2] asked whether a certain variant of this implication, which would in some situations be stronger, is also true. Namely, if we demand from  $G$  only that any  $X$  as above spans at least  $\frac{d}{2}|X|^2 + o(|V|^2)$  edges, so that locally  $G$  may have far more edges than a quasirandom graph would have, does it then still follow that one has at least  $(d^{|E(H)|} + o_H(1))|V(G)|^{|V(H)|}$  homomorphisms from  $H$  into  $G$ ?

To get these ideas more precise one may make the following

**Definition 1.1.** Let  $\varepsilon \in (0, 1)$  and  $d \in [0, 1]$  be given. A graph  $G$  on  $n$  vertices is said to be  $(\varepsilon, d)$ -dense if each  $X \subseteq V(G)$  with  $|X| \geq \varepsilon n$  spans at least  $\frac{d}{2}|X|^2$  edges.

Then what was asked in [2] is this:

**Question 1.2.** For which graphs  $H$  is it true that for each  $\delta > 0$  there exists an  $\varepsilon > 0$  such that there are at least  $(d^{|E(H)|} - \delta)|V(G)|^{|V(H)|}$  homomorphisms of  $H$  into any sufficiently large graph  $G$  that happens to be  $(\varepsilon, d)$ -dense for some real  $d$ ?

It was observed in [2] that the answer is affirmative if  $H$  is a clique, a complete multipartite graph, the line graph of a boolean cube, or a bipartite graph that satisfies SIDORENKO'S

---

*Key words and phrases.*  $(\varepsilon, d)$ -dense graphs, odd cycles, graph homomorphisms, Sidorenko's conjecture.

conjecture [3], which was formulated independently by ERDŐS and SIMONOVITS (see e.g. [4]). The last of these classes is known to contain all even cycles, and the authors of [2] wondered explicitly about the case of odd cycles. The main result of this article addresses this problem. In order to be able to state it more briefly, we introduce the following

**Definition 1.3.** Given a graph  $G$  and an integer  $r \geq 2$ , we write  $C_r(G)$  for the number of sequences  $(x_1, x_2, \dots, x_r) \in V(G)^r$  having the property that  $x_1x_2, x_2x_3, \dots, x_rx_1$  are edges of  $G$ .

**Theorem 1.4.** *If a graph  $G$  is  $(\varepsilon, d)$ -dense and possesses at least  $\frac{2}{\varepsilon - \varepsilon^2}$  vertices, then we have  $C_r(G) \geq (d^r - \varepsilon)|V(G)|^r$  for each odd number  $r \geq 3$ .*

The proof is prepared in the next section by some lemma that has an analytic flavour, whilst the main proof is deferred to Section 3. It uses an inequality due to G. R. BLAKLEY and P. A. ROY [1] that basically tells us that paths obey SIDORENKO'S conjecture.

**Lemma 1.5.** *For any positive integer  $k$  and any graph  $G$  satisfying  $|E(G)| \geq \frac{d}{2} |V(G)|^2$  there are at least  $d^k |V(G)|^{k+1}$  homomorphisms from a path of length  $k$  into  $G$ .*

## §2. MORE ON DENSENESS

In this section we prove that the property of being dense does more or less imply a weighted version of itself. A precise statement along those lines reads as follows:

**Lemma 2.1.** *Suppose that  $G$  is an  $(\varepsilon, d)$ -dense graph on  $n$  vertices. Then for every function  $f: V(G) \rightarrow [0, 1]$  satisfying  $\sum_{x \in V(G)} f(x) \geq \varepsilon n$  we have*

$$\sum_{xy \in E(G)} f(x)f(y) \geq \frac{d}{2} \left( \sum_{x \in V(G)} f(x) \right)^2 - n.$$

*Proof.* 1. The space of all functions  $f: V(G) \rightarrow [0, 1]$  satisfying  $\sum_{x \in V(G)} f(x) \geq \varepsilon n$  is compact and consequently it contains a member  $f_0$  for which the continuous expression

$$\sum_{xy \in E(G)} f(x)f(y) - \frac{d}{2} \left( \sum_{x \in V(G)} f(x) \right)^2$$

attains its least possible real value  $\Omega$ , and for which subject to this the set  $X$  of all  $x \in V(G)$  with  $f(x) \in \{0, 1\}$  has its maximal possible size. Evidently it suffices to prove  $\Omega \geq -n$ .

2. As a first step in this direction we will verify  $|V(G) - X| \leq 1$ . Assume contrariwise that there are two distinct vertices  $x$  and  $y$  belonging to  $V(G) - X$ . Let  $\eta$  be any real

number whose absolute value is so small that  $f_0(x) \pm \eta, f_0(y) \pm \eta \in [0, 1]$ . For any two real numbers  $a$  and  $b$  we write  $H(a, b)$  for the value that

$$\sum_{xy \in E(G)} f(x)f(y) - \frac{d}{2} \left( \sum_{x \in V(G)} f(x) \right)^2$$

attains for the function  $f: V(G) \rightarrow [0, 1]$  given by

$$f(z) = \begin{cases} a & \text{if } z = x \\ b & \text{if } z = y \\ f_0(z) & \text{if } z \neq x, y. \end{cases}$$

So for example  $H(f_0(x), f_0(y)) = \Omega$ . If  $xy$  were an edge of  $G$ , then there would have to exist real numbers  $A, B, C$ , and  $T$  not depending on  $\eta$  such that

$$\begin{aligned} H(f_0(x) + \eta, f_0(y) - \eta) &= (f_0(x) + \eta)(f_0(y) - \eta) + A(f_0(x) + \eta) + B(f_0(y) - \eta) + C \\ &= H(f_0(x), f_0(y)) + (f_0(y) - f_0(x) + A - B)\eta - \eta^2 \\ &= \Omega + T\eta - \eta^2. \end{aligned}$$

In each of the three cases  $T > 0$ ,  $T = 0$ , and  $T < 0$  it is easy to choose  $\eta$  in such a way that  $H(f_0(x) + \eta, f_0(y) - \eta) < \Omega$  holds. As this contradicts the supposed minimality of  $f_0$ , we have thereby shown that  $xy$  cannot be an edge of  $G$ . This, however, means that there exist three real numbers  $A, B$ , and  $C$  not depending on  $\eta$  such that

$$\begin{aligned} H(f_0(x) + \eta, f_0(y) - \eta) &= A(f_0(x) + \eta) + B(f_0(y) - \eta) + C \\ &= H(f_0(x), f_0(y)) + (A - B)\eta. \end{aligned}$$

Now the same contradiction arises as before unless  $A = B$ , in which case we have

$$H(f_0(x) + \eta, f_0(y) - \eta) = H(f_0(x), f_0(y))$$

for any  $\eta$ . Thus taking

$$\eta = \min(1 - f_0(x), f_0(y))$$

we get a contradiction to the extremal choice of  $X$ . We have thereby learned that there is indeed some vertex  $z \in V(G)$  with  $V(G) - \{z\} \subseteq X$ .

3. Let us now define  $\delta = f_0(z)$  and  $A = \{x \in X \mid f_0(x) = 1\} \cup \{z\}$ . Evidently we have

$$|A| \geq \sum_{x \in V(G)} f_0(x) \geq \varepsilon n$$

and, since  $G$  is  $(\varepsilon, d)$ -dense by hypothesis, this implies  $e(A) \geq \frac{1}{2} \cdot d |A|^2$ , where  $e(A)$  refers to the number of edges spanned by  $A$ . Notice that

$$\sum_{x \in V(G)} f_0(x) = |A| - (1 - \delta)$$

and

$$\sum_{xy \in E(G)} f_0(x)f_0(y) = e(A - \{z\}) + \delta N = e(A) - (1 - \delta)N,$$

where  $N$  denotes the number of edges from  $z$  to  $A - \{z\}$ . Thereby we obtain indeed

$$\begin{aligned} \Omega &= e(A) - (1 - \delta)N - \frac{d}{2}(|A| - (1 - \delta))^2 \\ &= (e(A) - \frac{d}{2}|A|^2) - (1 - \delta)(N - d|A|) - \frac{d}{2}(1 - \delta)^2 \\ &\geq -N - 1 \geq -n, \end{aligned}$$

which concludes the proof.  $\square$

### §3. THE PROOF OF THEOREM 1.4

In this section we shall finally prove Theorem 1.4. Write  $r = 2m + 1$  with some positive integer  $m$ . Given two vertices  $x$  and  $y$  of  $G = (V, E)$  we denote the number of sequences  $(a_0, \dots, a_m) \in V^{m+1}$  satisfying  $a_0 = x$ ,  $a_0 a_1, \dots, a_{m-1} a_m \in E$ , and  $a_m = y$  by  $q(x, y)$ . Clearly we have  $q(x, y) \leq n^{m-1}$ , where  $n = |V|$ , and

$$C_r(G) = \sum_{(x,y,z) \in V^3; yz \in E} q(x, y)q(x, z).$$

Writing  $Z$  for the set of all  $x \in V$ , for which  $\sum_{y \in V} q(x, y) \geq \varepsilon n^m$  holds, we obtain, by Lemma 2.1,

$$\begin{aligned} C_r(G) &\geq 2n^{2m-2} \sum_{x \in Z} \sum_{yz \in E} \frac{q(x, y)}{n^{m-1}} \cdot \frac{q(x, z)}{n^{m-1}} \\ &\geq \sum_{x \in Z} d \left( \sum_{y \in V} q(x, y) \right)^2 - 2n^{2m-1} \\ &\geq d \sum_{x \in V} \left( \sum_{y \in V} q(x, y) \right)^2 - d\varepsilon^2 n^{2m+1} - 2n^{2m}. \end{aligned}$$

Due to  $n \geq \frac{2}{\varepsilon - \varepsilon^2}$  we have  $d\varepsilon^2 n + 2 \leq \varepsilon n$  and it follows that

$$C_r(G) \geq \frac{d}{n} \left( \sum_{x,y \in V} q(x, y) \right)^2 - \varepsilon n^{2m+1}.$$

In the light of Lemma 1.5 this entails

$$\begin{aligned} C_r(G) &\geq \frac{d}{n} \left( \left( \frac{2|E|}{n^2} \right)^m \cdot n^{m+1} \right)^2 - \varepsilon n^{2m+1} \\ &\geq d^{2m+1} n^{2m+1} - \varepsilon n^{2m+1} = (d^r - \varepsilon) n^r, \end{aligned}$$

thereby completing the proof of Theorem 1.4.  $\square$

**Remark 3.1.** It should be clear that this method of proof allows to decide Question 1.2 in a few more cases.

**Acknowledgement.** I would like to thank MATHIAS SCHACHT for bringing Question 1.2 to my attention.

## REFERENCES

- [1] G. R. Blakley and P. Roy, *A Hölder type inequality for symmetric matrices with nonnegative entries*, Proc. Amer. Math. Soc. **16** (1965), 1244–1245. MR0184950  $\uparrow$ 1
- [2] Y. Kohayakawa, B. Nagle, V. Rödl, and M. Schacht, *Weak hypergraph regularity and linear hypergraphs*, J. Combin. Theory Ser. B **100** (2010), no. 2, 151–160, DOI 10.1016/j.jctb.2009.05.005. MR2595699  $\uparrow$ 1, 1
- [3] A. Sidorenko, *A correlation inequality for bipartite graphs*, Graphs Combin. **9** (1993), no. 2, 201–204, DOI 10.1007/BF02988307. MR1225933  $\uparrow$ 1
- [4] M. Simonovits, *Extremal graph problems, degenerate extremal problems, and supersaturated graphs*, Progress in graph theory (Waterloo, Ont., 1982), Academic Press, Toronto, ON, 1984, pp. 419–437. MR776819  $\uparrow$ 1

FACHBEREICH MATHEMATIK, UNIVERSITÄT HAMBURG, D-20146 HAMBURG, GERMANY  
*E-mail address:* Christian.Reiher@uni-hamburg.de